The Ext–Group of Unitary Equivalence Classes of Unital Extensions

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Abstract Let $\mathcal{A}$ be a unital separable nuclear C^*–algebra which belongs to the bootstrap category $\mathcal{N}$ and $\mathcal{B}$ be a separable stable C^*–algebra. In this paper, we consider the group $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$ consisting of the unitary equivalence classes of unital extensions $\tau\colon\mathcal{A}\to Q(\mathcal{B})$. The relation between $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$ and $\mathrm{Ext}(\mathcal{A},\mathcal{B})$ is established. Using this relation, we show the half–exactness of $\mathrm{Ext}_u(\cdot,\mathcal{B})$ and the (UCT) for $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$. Furthermore, under certain conditions, we obtain the half–exactness and Bott periodicity of $\mathrm{Ext}_u(\mathcal{A},\cdot)$.

Keywords unital extension, multiplier algebra, Ext–group, quasi–unital $*$–homomorphism

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1 Introduction and Preliminaries

For a C^*–algebra $\mathcal{E}$, let $\mathrm{M}_n(\mathcal{E})$ (resp. $\mathcal{E}^{1\times n}$) denote the set of all $n\times n$ (resp. $1\times n$) matrices over $\mathcal{E}$. If $a=(a_1,\cdots,a_n)\in\mathcal{E}^{1\times n}$, we set $a^T=\begin{pmatrix}a_1^*\\ \vdots\\ a_n^*\end{pmatrix}$. Suppose $\mathcal{E}$ has unit 1. Denote by $U(\mathcal{E})$ (resp. $U_0(\mathcal{E})$) the unitary group of $\mathcal{E}$ (resp. the connected component of 1 in $U(\mathcal{E})$). The definitions of K–groups of $\mathcal{E}$ can be found in [1]. Throughout the paper, $\mathcal{A}$ is a separable unclear C^*–algebra with unit $1_{\mathcal{A}}$ and $\mathcal{B}$ is a separable stable C^*–algebra.

Let $M(\mathcal{B})$ be the multiplier of $\mathcal{B}$ and set $Q(\mathcal{B})=M(\mathcal{B})/\mathcal{B}$. Let $\pi\colon M(\mathcal{B})\to Q(\mathcal{B})$ be the quotient map. It is well–known that the C^*–algebra extension of $\mathcal{A}$ by $\mathcal{B}$ can be identified with $\mathrm{Hom}(A,Q(\mathcal{B}))$ (the set of all $*$–homomorphisms from $\mathcal{A}$ to $Q(\mathcal{B})$). $\tau\in\mathrm{Hom}(A,Q(\mathcal{B}))$ is called to be unital extension if $\tau(1_{\mathcal{B}})=1_{Q(\mathcal{B})}$. Let $\mathrm{Hom}_1(A,Q(\mathcal{B}))$ be the set of all unital extensions. $\tau\in\mathrm{Hom}(A,Q(\mathcal{B}))$ (resp. $\mathrm{Hom}_1(A,Q(\mathcal{B}))$) is trivial, if there is a $*$–(resp. unital) homomorphism $\phi\colon\mathcal{A}\to M(\mathcal{B})$ such that $\tau=\pi\circ\phi$.

Two extensions $\tau_1,\tau_2\in\mathrm{Hom}(A,Q(\mathcal{B}))$ are unitarily equivalent (denoted by $\tau_1\sim_u\tau_2$) if there is a unitary $u\in M(\mathcal{B})$ such that $Ad_{\pi(u)}\circ\tau_=\tau_2$. Let $[\tau]$ (resp. $[\tau]_u$) denote the unitary equivalence of τ in $\mathrm{Hom}(A,Q(\mathcal{B}))$ (resp. $\mathrm{Hom}_1(A,Q(\mathcal{B}))$) and set $\mathrm{Ext}(\mathcal{A},\mathcal{B})=\{[\tau]|\,\tau\in\mathrm{Hom}(A,Q(\mathcal{B}))\}$, $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})=\{\tau]_u|\,\tau\in\mathrm{Hom}_1(A,Q(\mathcal{B}))\}\}$.

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Since $\mathcal{B}$ is table, there are isometries $u_1, \cdots, u_n \in M(\mathcal{B})$ such that $\sum_{i=1}^{n} u_i u_i^* = 1_{M(\mathcal{B})}$ (here $n = 1, 2, \cdots$). Let $\tau_1, \tau_2 \in \mathrm{Hom}(A, Q(\mathcal{B}))$ (or $\mathrm{Hom}_1(A, Q(\mathcal{B}))$). Define $\tau_1 \oplus \tau_2 \in \mathrm{Hom}(A, Q(\mathcal{B}))$ (or $\mathrm{Hom}_1(A, Q(\mathcal{B}))$) by

$$(\tau_1 \oplus \tau_2)(a) = (\pi(u_1), \pi(u_2))\,\mathrm{diag}\,(\tau_1(a), \tau_2(a))(\pi(u_1), \pi(u_2))^T,\ \forall\, a \in \mathcal{A}.$$

$[\tau_1 \oplus \tau_2]$ (or $[\tau_1 \oplus \tau_2]_2$) is independent of the choice of u_1, u_2 and equivalence classes of $[\tau_i]$ (or $[\tau_i]_u$), $i = 1, 2$. So we can define an addition in $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ (resp. $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$) by $[\tau_1] + [\tau_2] = [\tau_1 \oplus \tau_2]$ (resp. $[\tau_1]_u + [\tau_2] = [\tau_1 \oplus \tau_2]_u$).

Let $\tau_1, \tau_2 \in \mathrm{Hom}(A, Q(\mathcal{B}))$ (resp. $\mathrm{Hom}_1(A, Q(\mathcal{B}))$). $[\tau_1] = [\tau_2]$ (resp. $[\tau_1]_u = [\tau_2]_u$) means that there are (resp. unital) trivial extensions τ_0, τ_0' such that $\tau_1 \oplus \tau_0 \sim_u \tau_2 \oplus \tau_0'$. In this case, $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ become Abelian groups by [1, Corollary 15.8.4].

$\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ and $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ are different in general. $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ has Bott periodicity and six–term exact sequences for variables $\mathcal{A}$ or $\mathcal{B}$ and the universal coefficient formula for $\mathcal{A}$ and $\mathcal{B}$ etc. $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ has no these properties in general. But there are some relations between them. For examples, L.G. Brown and M. Dadarlat showed the following sequences

$$0 \longrightarrow \mathbb{Z}/\{h([1_\mathcal{A}])|\, h \in \mathrm{Hom}(K_0(\mathcal{A}), \mathbb{Z})\} \longrightarrow \mathrm{Ext}_u(\mathcal{A}, \mathcal{K}) \longrightarrow \mathrm{Ext}(\mathcal{A}, \mathcal{K}) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Ext}(K_0(\mathcal{A}), [1_\mathcal{A}], \mathbb{Z}) \longrightarrow \mathrm{Ext}_u(\mathcal{A}, \mathcal{K}) \longrightarrow \mathrm{Hom}(K_0(\mathcal{A}), \mathcal{K}) \longrightarrow 0$$

when $\mathcal{A}$ is in the bootstrap category $\mathcal{N}$ (cf. [2, Proposition 1, Theorem 2]); V. Manuilov and K. Thomsen in [3] presented the six–term exact sequence for $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ as follows

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \longrightarrow & \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}) & \longrightarrow & \mathrm{Ext}(\mathcal{A}, \mathcal{B}) \\ \uparrow & & & & \downarrow \\ \mathrm{Ext}(\mathcal{A}, S\mathcal{B}) & \longleftarrow & \mathrm{Ext}_u(\mathcal{A}, S\mathcal{B}) & \longleftarrow & K_1(\mathcal{B}) \end{array}$$

and H. Lin characterized the strongly unitary equivalence of two full essential extensions in [4] for $\mathcal{A} \in \mathcal{N}$ by means of the subgroup $H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))$ of $K_0(\mathcal{B})$ (see below).

In spirited by above results, we will use the subgroups

$$H_1(K_0(\mathcal{A}), K_i(\mathcal{B})) = \{h([1_\mathcal{A}])|\, h \in \mathrm{Hom}(K_0(\mathcal{A}), K_i(\mathcal{B}))\}$$

of $K_i(\mathcal{B})$, $i = 0, 1$, to give an exact sequence for $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ and $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ as follows

$$\begin{aligned} 0 \longrightarrow H_1(K_0(\mathcal{A}), K_0(\mathcal{B})) \longrightarrow & K_0(\mathcal{B}) \longrightarrow \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}) \longrightarrow \\ & \longrightarrow \mathrm{Ext}(\mathcal{A}, \mathcal{B}) \longrightarrow H_1(K_0(\mathcal{A}), K_1(\mathcal{B})) \longrightarrow 0 \end{aligned}$$

in this paper when $\mathcal{A} \in \mathcal{N}$. Using this exact sequence, we present the half–exactness of $\mathrm{Ext}_u(\cdot, \mathcal{B})$ and the (UCT) for $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$. Furthermore, under certain conditions, we obtain the half–exactness and Bott periodicity of $\mathrm{Ext}_u(\mathcal{A}, \cdot)$.

2 The main result

Let p, q be projections in the C^*–algebra $\mathcal{E}$. p and q are equivalent in $\mathcal{E}$, denoted by $p \sim q$ if there is $u \in \mathcal{E}$ such that $p = u^*u$ and $q = uu^*$. Since $\mathcal{B}$ is separable and stable, there are isometries $u_1, \cdots, u_n$ in $M(\mathcal{B})$ such that $\sum\limits_{i=1}^{n} u_i u_i^* = 1_{M(\mathcal{B})} = 1$, $\forall\, n \geq 2$. Thus we have $K_i(M(\mathcal{B})) \cong 0$, $i = 0, 1$ by [1] and $\mathrm{diag}\,(p, 1) \sim \mathrm{diag}\,(1, 1) = 1_2$ for any projection $p \in M(\mathcal{B})$. Moreover, the index map $\partial_0 \colon K_1(Q(\mathcal{B})) \to K_0(\mathcal{B})$ and the exponential map $\partial_1 \colon K_0(Q(\mathcal{B})) \to K_1(\mathcal{B})$ given in [1] are isomorphic.

By [5] or [6], there is a trivial extension $\tau_{\mathcal{A},\mathcal{B}} \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}))$ such that $\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_0 \sim_u \tau_{\mathcal{A},\mathcal{B}}$ for any unital trivial extension τ_0, i.e., $\tau_{\mathcal{A},\mathcal{B}}$ is a unital absorbing trivial extension. Thus, $[\tau_1]_u = [\tau_2]_u$ in $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ iff $\tau_1 \oplus \tau_{\mathcal{A},\mathcal{B}} \sim_u \tau_2 \oplus \tau_{\mathcal{A},\mathcal{B}}$.

Lemma 2.1. *Let $\mathcal{E}$ be a C^*–algebra with unit 1. Assume that there are isometries $v_1, \cdots, v_n \in \mathcal{E}$ with $\sum\limits_{i=1}^{n} v_i v_i^* = 1$. Let p a projection in $\mathrm{M}_n(\mathcal{E})$ and u be a unitary in $\mathrm{M}_n(\mathcal{E})$. Set*

$$q = (v_1, \cdots, v_n) p (v_1, \cdots, v_n)^T, \;\; w = (v_1, \cdots, v_n) u (v_1, \cdots, v_n)^T.$$

Then q is a projection in $\mathcal{E}$ and w is unitary in $\mathcal{E}$. Furthermore, $[q] = [p]$ in $K_0(\mathcal{E})$ and $[w] = [u]$ in $K_1(\mathcal{E})$.

Proof. It is easy to check that q is a projection and w is unitary in $\mathcal{E}$. Set $X = \begin{pmatrix} v_1 & \cdots & v_n \\ 0 & \cdots & \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} p$.

Then $X^*X = p$ and $XX^* = \mathrm{diag}\,(q, 0) \in \mathrm{M}_n(\mathcal{E})$.

The rest comes from [7, Corollary] or the proof of Lemma 3.1 in [8]. □

Lemma 2.2. *Let $\tau \in \mathrm{Hom}(\mathcal{A}, Q(\mathcal{B}))$ be a nonunital extension. If $[\tau(1_{\mathcal{A}})] = 0$ in $K_0(Q(\mathcal{B}))$, then there is a unital extension τ_0 such that $\tau \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_0 \oplus 0$. when τ is trivial, τ_0 can be chosen as a trivial extension.*

Proof. Put $q = \tau(1_{\mathcal{A}})$. Let u_1, u_2 be isometries in $M(\mathcal{B})$ with $u_1u_1^* + u_2u_2^* = 1$. Since $[q] = 0$ in $K_0(Q(\mathcal{B}))$, we can find $v \in \mathrm{M}_2(Q(\mathcal{B}))$ such that $vv^* = 1_2$, $v^*v = \mathrm{diag}\,(q, 1)$. Set $V_0 = (\pi(u_1), \pi(u_2)) v (\pi(u_1), \pi(u_2))^T$ and $\tau_0 = Ad_{V_0} \circ (\tau \oplus \tau_{\mathcal{A},\mathcal{B}})$. Then $V_0V_0^* = 1$ and $\tau_0(1_{\mathcal{A}}) = 1$. Note that for any $a \in \mathcal{A}$,

$$\begin{pmatrix} V_0 & 0 \\ 1 - V_0^*V_0 & V_0^* \end{pmatrix} \mathrm{diag}\,((\tau \oplus \tau_{\mathcal{A},\mathcal{B}})(a), 0) \begin{pmatrix} V_0^* & 1 - V_0^*V_0 \\ 0 & V_0 \end{pmatrix} = \mathrm{diag}\,(\tau_0(a), 0).$$

The assertion follows.

When τ is trivial, there is a nonunital $*$–homomorphism $\phi \colon \mathcal{A} \to M(\mathcal{B})$ such that $\tau = \pi \circ \phi$. Let $p = \phi(1_{\mathcal{A}})$. Since $[p] = 0$, there is $u \in M(\mathcal{B})$ such that $uu^* = 1_2$ and $u^*u = \mathrm{diag}\,(p, 1)$. Let $\psi \colon \mathcal{A} \to M(\mathcal{B})$ be a unital $*$–homomorphism such that $\tau_{\mathcal{A},\mathcal{B}} = \pi \circ \psi$. Set $U_0 = (u_1, u_2) u (u_1, u_2)^T$ and

$$\psi_0(a) = U_0 (u_1, u_2)\, \mathrm{diag}\,(\phi(a), \psi(a)) (u_1, u_2)^T U_0^*, \;\; \forall\, a \in \mathcal{A}.$$

Then $\tau_0 = \pi \circ \psi_0$ is a unital trivial extension and $\tau \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_0 \oplus 0$. □

Define the map $\Phi_{\mathcal{A},\mathcal{B}}\colon K_0(\mathcal{B}) \to \mathrm{Ext}_u(\mathcal{A},\mathcal{B})$ by $\Phi_{\mathcal{A},\mathcal{B}}(x) = [Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u$, where $v \in U(Q(\mathcal{B}))$ such that $\partial_0([v]) = x$.

Lemma 2.3. *Keeping symbols as above, we have $\Phi_{\mathcal{A},\mathcal{B}}$ is a homomorphism and moreover,*

(1) $\mathrm{Ker}\,\Phi_{\mathcal{A},\mathcal{B}} \subset H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))$;

(2) $\Phi_{\mathcal{A},\mathcal{B}} = [Ad_v \circ \tau_0]_u$ *for any unital trivial extension* τ_0.

Proof. We first prove that $\Phi_{\mathcal{A},\mathcal{B}}$ is well–defined. If there is $v' \in U(Q(\mathcal{B}))$ such that $\partial_0([v]) = \partial_0([v']) = x$, then $v_0 = v'v^* \in U_0(Q(\mathcal{B}))$ and hence there exists $u_0 \in U_0(M(\mathcal{B}))$ such that $\pi(u_0) = v_0$. Thus, $Ad_v \circ \tau_{\mathcal{A},\mathcal{B}} \sim_u Ad_{v'} \circ \tau_{\mathcal{A},\mathcal{B}}$.

Now let $x_1, x_2 \in K_0(\mathcal{B})$ and choose $v_1, v_2 \in U(Q(\mathcal{B}))$ such that $\partial_0([v_i]) = x_i$, $i = 1, 2$. Then $\partial_0([\,\mathrm{diag}\,(v_1, v_2)]) = x_1 + x_2$. Let u_1, u_2 be isometries in $M(\mathcal{B})$ such that $u_1u_1^* + u_2u_2^* = 1$. Set $z = (\pi(u_1), \pi(u_2))\,\mathrm{diag}\,(v_1, v_2)\pi(u_1), \pi(u_2))^T$. Then $[z] = [\,\mathrm{diag}\,(v_1, v_2)]$ in $K_1(\mathcal{B})$ by Lemma 2.1 and $\partial_0([z]) = x_1 + x_2$. Moreover, we have

$$[Ad_z \circ (\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}})]_u = [(Ad_{v_1} \circ \tau_{\mathcal{A},\mathcal{B}}) \oplus (Ad_{v_2} \circ \tau_{\mathcal{A},\mathcal{B}})]_u = [Ad_{v_1} \circ \tau_{\mathcal{A},\mathcal{B}}]_u + [Ad_{v_2} \circ \tau_{\mathcal{A},\mathcal{B}}]_u$$

in $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$. Let $w_0 \in U(M(\mathcal{B}))$ such that $\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}} = Ad_{\pi(w_0)} \circ \tau_{\mathcal{A},\mathcal{B}}$ and put $z_0 = \pi(w_0^*)z\pi(w_0)$. Then $[z_0] = [z]$ in $K_1(Q(\mathcal{B}))$ and

$$\begin{aligned}\Phi_{\mathcal{A},\mathcal{B}}(x_1 + x_2) =& [Ad_{z_0} \circ \tau_{\mathcal{A},\mathcal{B}}]_u = [Ad_z \circ (\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}}]_u\\ =& [Ad_{v_1} \circ \tau_{\mathcal{A},\mathcal{B}}]_u + [Ad_{v_2} \circ \tau_{\mathcal{A},\mathcal{B}}]_u\\ =& \Phi_{\mathcal{A},\mathcal{B}}(x_1) + \Phi_{\mathcal{A},\mathcal{B}}(x_2).\end{aligned}$$

When $x = 0$ in $K_0(\mathcal{B})$, there is $u \in U(M(\mathcal{B}))$ such that $v = \pi(u)$ and consequently, $[Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u = 0$ in $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$.

Let $e \in \mathrm{Ker}\,\Phi_{\mathcal{A},\mathcal{B}}$. Then there is $v \in U(Q(\mathcal{B}))$ such that $\partial_0([v]) = e$ and $[Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u = 0$ in $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$. Thus, we can find $u \in U(M(\mathcal{B}))$ such that $(Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}) \oplus \tau_{\mathcal{A},\mathcal{B}} = Ad_{\pi(u)} \circ \tau_{\mathcal{A},\mathcal{B}}$. Set

$$\tilde{v} = (\pi(u_1), \pi(u_2))\,\mathrm{diag}\,(v, 1)(\pi(u_1), \pi(u_2))^T.$$

Then $[\tilde{v}] = [v]$ in $K_1(\mathcal{B})$ by Lemma 2.1 and

$$Ad_{\pi(u)} \circ \tau_{\mathcal{A},\mathcal{B}} = Ad_{\tilde{v}} \circ (\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}}) = Ad_{\tilde{v}} \circ Ad_{\pi(w_0)} \circ \tau_{\mathcal{A},\mathcal{B}}.$$

Set $\hat{v} = \pi(u^*)\tilde{v}\pi(w_0)$. Then $[\hat{v}] = [v]$ and $Ad_{\hat{v}} \circ \tau_{\mathcal{A},\mathcal{B}} = \tau_{\mathcal{A},\mathcal{B}}$. Therefore, we can define a unital extension $\hat{\tau}\colon C(\mathbf{S}^1, \mathcal{A}) \to Q(\mathcal{B})$ by $\hat{\tau}(a) = \tau_{\mathcal{A},\mathcal{B}}(a)$ for $a \in \mathcal{A}$ and $\hat{\tau}(z1_{\mathcal{A}}) = \hat{v}$, $z \in \mathbf{S}^1$.

Let $\theta_{\mathcal{A}}\colon K_0(\mathcal{A}) \to K_1(S\mathcal{A})$ be the Bott map given in [1, Definition 9.1.1] and let $i_{\mathcal{A}}\colon K_1(S\mathcal{A}) \to K_1(C(\mathbf{S}^1, \mathcal{A}))$ be the inclusion. Then $h = \partial_0 \circ \hat{\tau}_*^1 \circ i_{\mathcal{A}} \circ \theta_{\mathcal{A}} \in \mathrm{Hom}(K_0(\mathcal{A}), K_0(\mathcal{B}))$ and $h([1_{\mathcal{A}}]) = \partial_0([\hat{v}]) = e$, where $\hat{\tau}_*^1$ is the induced map of $\hat{\tau}$ on $K_1(C(\mathbf{S}^1, \mathcal{A}))$. This proves (1).

(2) Let $w \in U(\mathrm{M}_2(\mathcal{B}))$ such that $\pi_2(w) = \mathrm{diag}\,(v, v^*)$, where π_k is the induced homomorphism of π on $\mathrm{M}_k(\mathcal{B})$. Put

$$w_1 = (u_1, u_2)w(u_1, u_2)^T \text{ and } v_0 = (\pi(u_1), \pi(u_2))\,\mathrm{diag}\,(v_1, v_2)(\pi(u_1), \pi(u_2))^T.$$

Then $\pi(w_1) = v_0$ and

$$\begin{aligned} Ad_v \circ \tau_0 \oplus Ad_{v^*} \circ \tau_{\mathcal{A},\mathcal{B}} =& Ad_{\pi(w_1)} \circ (\tau_0 \oplus \tau_{\mathcal{A},\mathcal{B}}) \\ Ad_v \circ \tau_{\mathcal{A},\mathcal{B}} \oplus Ad_{v^*} \circ \tau_{\mathcal{A},\mathcal{B}} =& Ad_{\pi(w_1)} \circ (\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}}). \end{aligned}$$

Thus, $[Ad_v \circ \tau_0]_u = [Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u$ in $\mathrm{Ext}_u(\mathcal{A},\mathcal{B})$. □

Let $\mathcal{N}$ be the bootstrap category defined in [1] or [9]. Then for any $\mathcal{A} \in \mathcal{N}$, we have following exact sequence (UCT) (cf. [10]):

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(K_*(\mathcal{A}), K_*(\mathcal{B})) \xrightarrow{\kappa^{-1}} \mathrm{Ext}(\mathcal{A},\mathcal{B}) \xrightarrow{\Gamma_{\mathcal{A},\mathcal{B}}} \mathrm{Hom}(K_0(\mathcal{A}), K_1(\mathcal{B})) \oplus \mathrm{Hom}(K_1(\mathcal{A}), K_0(\mathcal{B})) \to 0$$

where $\Gamma_{\mathcal{A},\mathcal{B}}([\tau]) = (\partial_1 \circ \tau_*^0, \partial_0 \circ \tau_*^1)$ and τ_*^i is the induced homomorphism of τ on $K_i(\mathcal{A})$, $i = 0, 1$, κ is a bijective natural map from $\mathrm{Ker}\,\Gamma_{\mathcal{A},\mathcal{B}}$ onto $\mathrm{Ext}_{\mathbb{Z}}(K_*(\mathcal{A}), K_*(\mathcal{B})) = \mathrm{Ext}_{\mathbb{Z}}(K_0(\mathcal{A}), K_0(\mathcal{B})) \oplus \mathrm{Ext}_{\mathbb{Z}}(K_1(\mathcal{A}), K_1(\mathcal{B}))$.

By means of (UCT), we can obtain our main result in the paper as follows.

Theorem 2.4. *Suppose that $\mathcal{A}$ is a separable nuclear unital C^*–algebra and $\mathcal{B}$ is a separable C^*–algebra. If $\mathcal{A} \in \mathcal{N}$, then we have following exact sequence of groups:*

$$\begin{aligned} 0 \longrightarrow H_1(K_0(\mathcal{A}), K_0(\mathcal{B})) \xrightarrow{j_{\mathcal{B}}} K_0(\mathcal{B}) \xrightarrow{\Phi_{\mathcal{A},\mathcal{B}}} \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \longrightarrow& \\ \xrightarrow{i_{\mathcal{A},\mathcal{B}}} \mathrm{Ext}(\mathcal{A},\mathcal{B}) \xrightarrow{\rho_{\mathcal{A},\mathcal{B}}} H_1(K_0(\mathcal{A}), K_1(\mathcal{B})) \longrightarrow 0&, \end{aligned}$$

where, $j_{\mathcal{B}}$ is an inclusion, $i_{\mathcal{A},\mathcal{B}}([\tau]_u) = [\tau]$, $\forall\, \tau \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}))$ and $\rho_{\mathcal{A},\mathcal{B}}([\tau]) = \partial_1([\tau(1_{\mathcal{A}})])$, $\forall\, \tau \in \mathrm{Hom}(\mathcal{A}, Q(\mathcal{B}))$.

Proof. Let $[u] \in H_1(K_0(\mathcal{A}), K_1(\mathcal{B}))$. Then there is $h \in \mathrm{Hom}(K_0(\mathcal{A}), K_1(\mathcal{B}))$ such that $[u] = h([1_{\mathcal{A}}]) \in K_1(\mathcal{B})$. By (UCT), there is $\tau \in \mathrm{Hom}(\mathcal{A}, Q(\mathcal{B}))$ such that $\Gamma_{\mathcal{A},\mathcal{B}}([\tau]) = (h, 0)$. Therefore,

$$[u] = h([1_{\mathcal{A}}]) = \partial_1(\tau_*^0([1_{\mathcal{A}}])) = \partial_1([\tau(1_{\mathcal{A}})]) = \rho_{\mathcal{A},\mathcal{B}}([\tau]),$$

that is, $\rho_{\mathcal{A},\mathcal{B}}$ is surjective.

Since $[1_{M(\mathcal{B})}] = 0$ in $K_0(M(\mathcal{B}))$ implies that $[1_{Q(\mathcal{B})}] = 0$ in $K_0(Q(\mathcal{B}))$, we have $\rho_{\mathcal{A},\mathcal{B}}([\tau]) = 0$ when τ is a unital extension and hence $\mathrm{Ran}\,(i_{\mathcal{A},\mathcal{B}}) \subset \mathrm{Ker}\,\rho_{\mathcal{A},\mathcal{B}}$. Now let $[\tau] \in \mathrm{Ker}\,\rho_{\mathcal{A},\mathcal{B}}$ and put $q = \tau(1_{\mathcal{A}})$. Then $\partial_1([q]) = 0$ and hence by Lemma 2.2, there is a unital extension τ_0 such that $\tau \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_0 \oplus 0$. Thus, $[\tau] = [\tau_0]$ in $\mathrm{Ext}(\mathcal{A},\mathcal{B})$, i.e., $\mathrm{Ker}\,\rho_{\mathcal{A},\mathcal{B}} \subset \mathrm{Ran}\,(i_{\mathcal{A},\mathcal{B}})$.

Let $v \in U(Q(\mathcal{B}))$. we can pick $u \in U(\mathrm{M}_2(M(\mathcal{B})))$ such that $\mathrm{diag}\,(v, v^*) = \pi_2(u)$. It follows from $\pi_2(u)\,\mathrm{diag}\,(\tau_{\mathcal{A},\mathcal{B}}, 0)\pi_2(u^*) = \mathrm{diag}\,(Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}, 0)$ that $\mathrm{Ran}\,(\Phi_{\mathcal{A},\mathcal{B}}) \subset \mathrm{Ker}\,i_{\mathcal{A},\mathcal{B}}$. On the other hand, let τ be a unital extension such that $\tau \oplus \tau_1 \sim_u \tau_2$ for some trivial extensions τ_1 and τ_2. If τ_1, τ_2 are all unital, then $[\tau]_u = 0$; if τ_1, τ_2 are all non–unital, then by Lemma 2.2, we can find unital trivial extensions τ_1' and τ_2' such that $\tau_i \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_i' \oplus 0$, $i = 1, 2$. Consequently, $\tau \oplus \tau_1' \oplus 0 \sim_u \tau_2' \oplus 0$ and hence

$$\tau \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau \oplus \tau_1' \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_2' \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_{\mathcal{A},\mathcal{B}} \oplus 0 \sim_u \tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0,$$

i.e., there is $u \in U(M(\mathcal{B}))$ such that $\tau \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0 = Ad_{\pi(u)} \circ (\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}} \oplus 0)$. Let u_1, u_2 be isometries in $M(\mathcal{B})$ such that $u_1u_1^* + u_2u_2^* = 1$. Set $w = (u_1, u_2)^T u(u_1, u_2) \in U(\mathrm{M}_2(M(\mathcal{B})))$. Then

$$\mathrm{diag}\,(\tau \oplus \tau_{\mathcal{A},\mathcal{B}}(a), 0) = \pi_2(w)\,\mathrm{diag}\,(\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}}(a), 0)\pi_2(w)^*,\ \forall\, a \in \mathcal{A}.$$

It follows that $\pi_2(w)$ has the form $\pi_2(w) = \mathrm{diag}\,(v_1, v_2)$ and hence $\tau\oplus\tau_{\mathcal{A},\mathcal{B}} = Ad_{v_1}\circ(\tau_{\mathcal{A},\mathcal{B}}\oplus\tau_{\mathcal{A},\mathcal{B}})$. Let $w_0 \in U(M(\mathcal{B}))$ such that $\tau_{\mathcal{A},\mathcal{B}} \oplus \tau_{\mathcal{A},\mathcal{B}} = Ad_{\pi(w_0)} \circ \tau_{\mathcal{A},\mathcal{B}}$. Put $x = \partial_0([v_1\pi(w_0)])$. Then $\Phi_{\mathcal{A},\mathcal{B}}(x) = [Ad_{v_1\pi(w_0)} \circ \tau_{\mathcal{A},\mathcal{B}}]_u = [\tau \oplus \tau_{\mathcal{A},\mathcal{B}}]_u = [\tau]_u$.

By Lemma 2.3 (1), $\mathrm{Ker}\,\Phi_{\mathcal{A},\mathcal{B}} \subset \mathrm{Ran}\,(j_{\mathcal{B}})$. We now prove $\mathrm{Ran}\,(j_{\mathcal{B}}) \subset \mathrm{Ker}\,\Phi_{\mathcal{A},\mathcal{B}}$.

Let $x \in H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))$. Then there is $h \in \mathrm{Hom}(K_0(\mathcal{A}), K_0(\mathcal{B}))$ such that $x = h([1_{\mathcal{A}}])$. Let $p_{\mathcal{A}}\colon K_1(C(\mathbf{S}^1, \mathcal{A})) \to K_1(S\mathcal{A})$ be the projective map. Set $h_0 = h \circ \theta_{\mathcal{A}}^{-1} \circ p_{\mathcal{A}}$. Then $h_0 \in \mathrm{Hom}(K_1(C(\mathbf{S}^1, \mathcal{A})), K_0(\mathcal{B}))$ with $h_0([z1_{\mathcal{A}}]) = h \circ \theta_A^{-1}([z1_{\mathcal{A}}]) = h([1_{\mathcal{A}}]) = x$. Thus, by (UCT), there is $\tilde{\tau} \in \mathrm{Hom}(C(\mathbf{S}^1, \mathcal{A}), Q(\mathcal{B}))$ such that $\Gamma_{C(\mathbf{S}^1,\mathcal{A}),\mathcal{B}}([\tilde{\tau}]) = (0, h_0)$, i.e., $\partial_0 \circ \hat{\tau}_*^1 = h_0$ and $\partial_1 \circ \hat{\tau}_*^0 = 0$. So $[\tilde{\tau}(1_{\mathcal{A}})] = 0$ in $K_0(Q(\mathcal{B}))$. In this case, $\tilde{\tau}$ can be chosen as the unital one by Lemma 2.2. Set $v = \tilde{\tau}(z1_{\mathcal{A}})$ and let τ be the restriction of $\tilde{\tau}$ on $\mathcal{A}$. Then $x = \partial_0([v])$ and $v\tau(a) = \tau(a)v$, $\forall\, a \in \mathcal{A}$. Pick $w \in U(\mathrm{M}_2(M(\mathcal{B})))$ such that $\pi_2(w) = \mathrm{diag}\,(v, v^*)$. Note that

$$\pi_2(w)\,\mathrm{diag}\,(\tau(a), \tau_{\mathcal{A},\mathcal{B}}(a))\pi_2(w)^* = \mathrm{diag}\,(v\tau(a)v^*, v^*\tau_{\mathcal{A},\mathcal{B}}(a)v),\ \forall\, a \in \mathcal{A}.$$

We have

$$[\tau]_u = [Ad_v \circ \tau]_u + [Ad_{v^*} \circ \tau_{\mathcal{A},\mathcal{B}}]_u = [\tau]_u + [Ad_{v^*} \circ \tau_{\mathcal{A},\mathcal{B}}]_u$$

which implies that $[Ad_{v^*} \circ \tau_{\mathcal{A},\mathcal{B}}]_u = 0$ in $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ since $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ is a group. So $\phi_{\mathcal{A},\mathcal{B}}(x) = [Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u = -[Ad_{v^*} \circ \tau_{\mathcal{A},\mathcal{B}}]_u = 0$. □

Let $x\in K_0(\mathcal{B})$. Write $[x]_{\mathcal{A},\mathcal{B}}$ to denote the equivalence class of x in $K_0(\mathcal{B})/H_1(K_0(\mathcal{A}),K_0(\mathcal{B}))$. Since $\mathrm{Ker}\,\Phi_{\mathcal{A},\mathcal{B}} = H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))$ by Theorem 2.4, we can define the homomorphism

$$\Phi'_{\mathcal{A},\mathcal{B}}\colon K_0(\mathcal{B})/H_1(K_0(\mathcal{A}), K_0(\mathcal{B})) \to \mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$$

by $\Phi'_{\mathcal{A},\mathcal{B}}([x]_{\mathcal{A},\mathcal{B}}) = \Phi_{\mathcal{A},\mathcal{B}}(x)$. Thus, we have

Corollary 2.5. *Let $\mathcal{A}$, $\mathcal{B}$, $i_{\mathcal{A},\mathcal{B}}$ and $\rho_{\mathcal{A},\mathcal{B}}$ be as in Theorem 2.4. Then we have following exact sequence of groups:*

$$0 \longrightarrow K_0(\mathcal{B})/H_1(K_0(\mathcal{A}), K_0(\mathcal{B})) \xrightarrow{\Phi'_{\mathcal{A},\mathcal{B}}} \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}) \xrightarrow{i_{\mathcal{A},\mathcal{B}}} \mathrm{Ext}(\mathcal{A}, \mathcal{B}) \xrightarrow{\rho_{\mathcal{A},\mathcal{B}}} H_1(K_0(\mathcal{A}), K_1(\mathcal{B})) \longrightarrow 0.$$

3 The half–exactness of $\mathrm{Ext}_u(\cdot, \mathcal{B})$ and the (UCT) for $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$

Let $\mathcal{J}$ be a closed ideal of the separable nuclear unital C^*–algebra $\mathcal{A}$ and let $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{J} = \mathcal{C}$ be the quotient map. Then $\mathcal{J}$ and $\mathcal{C}$ are nuclear. Define $q^*\colon \mathrm{Ext}(\mathcal{C}, \mathcal{B}) \to \mathrm{Ext}(\mathcal{A}, \mathcal{B})$ and $q_u^*\colon \mathrm{Ext}_u(\mathcal{C}, \mathcal{B}) \to \mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$ respectively, by $q^*([\tau]) = [\tau \circ q]$ and $q_u^*([\tau]_u) = [\tau \circ q]_u$. Let i^* (resp. i_u^*) be the induced homomorphism of the inclusion $i\colon \mathcal{J} \to \mathcal{A}$ on $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ (resp. $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$).

Proposition 3.1. *Let $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{C}$ be as above. Suppose that $\mathcal{A}$ and $\mathcal{C}$ are all in $\mathcal{N}$. Then*

$$\mathrm{Ext}_u(\mathcal{C},\mathcal{B}) \xrightarrow{q_u^*} \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \xrightarrow{i_u^*} \mathrm{Ext}(\mathcal{J},\mathcal{B}) \tag{3.1}$$

is exact in the middle (i.e., $\mathrm{Ran}\,(q_u^*) = \mathrm{Ker}\, i_u^*$*). In addition, if there exists a unital $*$–homomorphism $r\colon \mathcal{C} \to \mathcal{A}$ such that $q \circ r = \mathrm{id}_{\mathcal{C}}$, then* (3.1) is split exact.

Proof. It is clear that $H_1(K_0(\mathcal{C}), K_i(\mathcal{B})) \subset H_1(K_0(\mathcal{A}), K_i(\mathcal{B})) \subset K_i(\mathcal{B})$, $i = 0, 1$. Let

$$i_1\colon H_1(K_0(\mathcal{C}), K_1(\mathcal{B})) \to H_1(K_0(\mathcal{A}), K_1(\mathcal{B}))$$

be the inclusion and define

$$i_0\colon K_0(\mathcal{B})/H_1(K_0(\mathcal{C}), K_0(\mathcal{B})) \to K_0(\mathcal{B})/H_1(K_0(\mathcal{A}), K_0(\mathcal{B})$$

by $i_0([x]_{\mathcal{C},\mathcal{B}}) = [x]_{\mathcal{A},\mathcal{B}}$. Let $x \in K_0(\mathcal{B})$ and $v \in U(Q(\mathcal{B}))$ such that $\Phi_{\mathcal{A},\mathcal{B}}(x) = [Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u$. Since $\tau_{\mathcal{C},\mathcal{B}} \circ q \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}))$ is trivial extension, we have by Lemma 2.3 (2),

$$q_u^* \circ \Phi'_{\mathcal{C},\mathcal{B}}([x]_{\mathcal{C},\mathcal{B}}) = [Ad_v \circ \tau_{\mathcal{C},\mathcal{B}} \circ q]_u = [Ad_v \circ \tau_{\mathcal{A},\mathcal{B}}]_u = \Phi_{\mathcal{A},\mathcal{B}}(x) = \Phi'_{\mathcal{A},\mathcal{B}} \circ i_0([x]_{\mathcal{C},\mathcal{B}}).$$

It is easy to check that $q^* \circ i_{\mathcal{C},\mathcal{B}} = i_{\mathcal{A},\mathcal{B}} \circ q_u^*$, $i^* \circ i_{\mathcal{A},\mathcal{B}} = i_u^*$ and $i_1 \circ \rho_{\mathcal{C},\mathcal{B}} = \rho_{\mathcal{A},\mathcal{B}} \circ q^*$. So we get following commutative diagram of Abelian groups:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{C}), K_0(\mathcal{B}))} \arrow[r, "i_0"] \arrow[d, "\Phi'_{\mathcal{C},\mathcal{B}}"'] & \dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))} \arrow[d, "\Phi'_{\mathcal{A},\mathcal{B}}"'] & \\
\mathrm{Ext}_u(\mathcal{C},\mathcal{B}) \arrow[r, "q_u^*"] \arrow[d, "i_{\mathcal{C},\mathcal{B}}"'] & \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \arrow[r, "i_u^*"] \arrow[d, "i_{\mathcal{A},\mathcal{B}}"'] & \mathrm{Ext}(\mathcal{J},\mathcal{B}) \arrow[d, equal] \\
\mathrm{Ext}(\mathcal{C},\mathcal{B}) \arrow[r, "q^*"] \arrow[d, "\rho_{\mathcal{C},\mathcal{B}}"'] & \mathrm{Ext}(\mathcal{A},\mathcal{B}) \arrow[r, "i^*"] \arrow[d, "\rho_{\mathcal{A},\mathcal{B}}"'] & \mathrm{Ext}(\mathcal{J},\mathcal{B}) \\
H_1(K_0(\mathcal{C}), K_1(\mathcal{B})) \arrow[r, "i_1"] \arrow[d] & H_1(K_0(\mathcal{A}), K_1(\mathcal{B})) \arrow[d] & \\
0 & 0 &
\end{tikzcd} \tag{3.2}$$

in which $\mathrm{Ran}\,(q^*) = \mathrm{Ker}\, i^*$ by [1, Theorem 15.11.2] and two columns are exact by Corollary 2.5.

Since $q \circ i = 0$, we have $i_u^* \circ q_u^* = 0$ and hence $\mathrm{Ran}\,(q_u^*) \subset \mathrm{Ker}\, i_u^*$. Noting that i_0 is surjective and i_1 is injective, we can use the commutative diagram (3.2) to obtain that $\mathrm{Ker}\, i_u^* \subset \mathrm{Ran}\,(q_u^*)$. Thus, (3.1) is exact in the middle.

If the $*$–homomorphism $r\colon \mathcal{C} \to \mathcal{A}$ satisfies $q \circ r = \mathrm{id}_{\mathcal{C}}$, then $r_u^* \circ q_u^* = \mathrm{id}$ on $\mathrm{Ext}_u(\mathcal{C},\mathcal{B})$ and i_0, i_1 are all identity maps. So q_u^* is injective. Since i^* is surjective by [1, Theorem 15.11.2], there is $[\tau'] \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ such that $i^*([\tau']) = [\tau]$. Pick $[\tau''] \in \mathrm{Ext}(\mathcal{C},\mathcal{B})$ with $\rho_{\mathcal{C},\mathcal{B}}([\tau'']) = i_1^{-1} \circ \rho_{\mathcal{A},\mathcal{B}}([\tau'])$.

Using $\rho_{\mathcal{A},\mathcal{B}} \circ q^* = i_1 \circ \rho_{\mathcal{C},\mathcal{B}}$, we get that $\rho_{\mathcal{A},\mathcal{B}} \circ q^*([\tau'']) = \rho_{\mathcal{A},\mathcal{B}}([\tau'])$. Thus, there is $[\tau_0]_u \in \mathrm{Ext}_u(\mathcal{A},\mathcal{B})$ such that $i_{\mathcal{A},\mathcal{B}}([\tau_0]_u) = [\tau'] - q^*([\tau''])$. From $i^* \circ i_{\mathcal{A},\mathcal{B}} = i_u^*$, $\mathrm{Ran}\,(q^*) = \mathrm{Ker}\, i^*$, $i^*([\tau']) = [\tau]$, we get that $[\tau] = i_u^*([\tau_0])$, i.e., i_u^* is surjective. In this case, (3.1) becomes an exact sequence:

$$0 \longrightarrow \mathrm{Ext}_u(\mathcal{C},\mathcal{B}) \xrightarrow{q_u^*} \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \xrightarrow{i_u^*} \mathrm{Ext}(\mathcal{J},\mathcal{B}) \longrightarrow 0 \tag{3.3}$$

with $r_u^* \circ q_u^* = \mathrm{id}$. Define homomorphism $\rho\colon \mathrm{Ext}(\mathcal{J},\mathcal{B}) \to \mathrm{Ext}_u(\mathcal{A},\mathcal{B})$ by

$$\rho([\sigma]) = [\sigma']_u - q_u^* \circ r_u^*([\sigma']_u) \quad \text{with} \quad [\sigma] = i_u^*([\sigma']_u).$$

It is easy to check that ρ is well–defined and $i_u^* \circ \rho = \mathrm{id}$ on $\mathrm{Ext}(\mathcal{J},\mathcal{B})$. Thus, (3.3) is split exact. $\square$

Let $\mathrm{Ext}_{\mathbb{Z}}(K_0(\mathcal{A}),[1_{\mathcal{A}}],K_0(\mathcal{B}))$ be the set of isomorphism classes of all extensions of $K_0(\mathcal{A})$ by $K_0(\mathcal{B})$ with base point of the form

$$0 \longrightarrow K_0(\mathcal{B}) \longrightarrow (G,g_0) \xrightarrow{\phi} (K_0(\mathcal{A}),[1_{\mathcal{A}}]) \longrightarrow 0,$$

where $\phi(g_0) = [1_{\mathcal{A}}]$. The natural map from $\mathrm{Ext}_{\mathbb{Z}}(K_0(\mathcal{A}),[1_{\mathcal{A}}],K_0(\mathcal{B}))$ to $\mathrm{Ext}_{\mathbb{Z}}(K_0(\mathcal{A}),K_0(\mathcal{B}))$ has the kernel isomorphic to $K_0(\mathcal{B})/H_1(K_0(\mathcal{A}),K_0(\mathcal{B}))$ (cf. [2, P584]). Define the homomorphism

$$\bar{\Gamma}_{\mathcal{A},\mathcal{B}}\colon \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \to \mathrm{Hom}_0(K_0(\mathcal{A}),K_1(\mathcal{B})) \oplus \mathrm{Hom}(K_1(\mathcal{A}),K_0(\mathcal{B})),$$

by $\bar{\Gamma}_{\mathcal{A},\mathcal{B}}([\tau]_u) = (\partial_1 \circ \tau_*^0, \partial_0 \circ \tau_*^1)$ for $\partial_1 \circ \tau_*^0([1_{\mathcal{A}}]) = 0$ ($[1_{Q(\mathcal{B})}] = 0$ in $K_0(Q(\mathcal{B}))$), where $\mathrm{Hom}_0(K_0(\mathcal{A}),K_1(\mathcal{B})) = \{h \in \mathrm{Hom}(K_0(\mathcal{A}),K_1(\mathcal{B}) |\, h([1_{\mathcal{A}}]) = 0\}$.

Let $[\tau]_u \in \mathrm{Ker}\,\bar{\Gamma}_{\mathcal{A},\mathcal{B}}$. We have following isomorphism classes of extensions

$$0 \longrightarrow K_0(\mathcal{B}) \longrightarrow (K_0(E),[1_E]) \xrightarrow{\phi_*^0} (K_0(\mathcal{A}),[1_{\mathcal{A}}]) \longrightarrow 0$$
$$0 \longrightarrow K_1(\mathcal{B}) \longrightarrow K_1(E) \xrightarrow{\phi_*^1} K_1(\mathcal{A}) \longrightarrow 0,$$

where $E = \{(a,b) \in \mathcal{A} \oplus M(\mathcal{B}) |\, \pi(b) = \tau(a)\}$, $\phi(a,b) = a$, $\forall\,(a,b) \in E$. So there exists a natural map $\bar{\kappa}$ from $\mathrm{Ker}\,\bar{\Gamma}_{\mathcal{A},\mathcal{B}}$ to $\mathrm{Ext}_{\mathbb{Z}}(K_0(\mathcal{A}),[1_{\mathcal{A}}],K_0(\mathcal{B})) \oplus \mathrm{Ext}_{\mathbb{Z}}(K_1(\mathcal{A}),K_1(\mathcal{B})) \triangleq \mathrm{Ext}_{\mathbb{Z}}(K_*(\mathcal{A}),[1_{\mathcal{A}}],K_*(\mathcal{B}))$. Moveover, we have

Proposition 3.2. *Suppose that $\mathcal{A}$ is a separable nuclear unital C^*–algebra and $\mathcal{B}$ is a separable stable C^*–algebra. If $\mathcal{A} \in \mathcal{N}$, then $\bar{\kappa}$ is bijective and the following sequence of groups is exact:*

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}(K_*(\mathcal{A}),[1_{\mathcal{A}}],K_*(\mathcal{B})) \xrightarrow{\bar{\kappa}^{-1}} \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \xrightarrow{\bar{\Gamma}_{\mathcal{A},\mathcal{B}}}$$
$$\longrightarrow \mathrm{Hom}_0(K_0(\mathcal{A}),K_1(\mathcal{B})) \oplus \mathrm{Hom}(K_1(\mathcal{A}),K_0(\mathcal{B})) \longrightarrow 0.$$

Proof. Let $(h_0,h_1) \in \mathrm{Hom}_0(K_0(\mathcal{A}),K_1(\mathcal{B})) \oplus \mathrm{Hom}(K_1(\mathcal{A}),K_0(\mathcal{B}))$. Then we can find $[\tau] \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ such that $\Gamma_{\mathcal{A},\mathcal{B}}([\tau]) = (h_0,h_1)$ by (UCT). Since $h_0([1_{\mathcal{A}}]) = \partial_1 \circ \tau_*^0([1_{\mathcal{A}}]) = 0$, we can pick a unital extension τ_0 with $[\tau] = [\tau_0]$ in $\mathrm{Ext}(\mathcal{A},\mathcal{B})$ by Lemma 2.2. Thus,

$$\bar{\Gamma}_{\mathcal{A},\mathcal{B}}([\tau_0]_u) = \Gamma_{\mathcal{A},\mathcal{B}}([\tau_0]) = \Gamma_{\mathcal{A},\mathcal{B}}([\tau]) = (h_0,h_1),$$

that is, $\bar{\Gamma}_{\mathcal{A},\mathcal{B}}$ is surjective.

When $[\tau] \in \operatorname{Ker}\Gamma_{\mathcal{A},\mathcal{B}}$, $[\tau(1_{\mathcal{A}})] = 0$ in $K_0(Q(\mathcal{B}))$. So there is a unital extension τ_1 such that $[\tau_1] = [\tau]$ in $\operatorname{Ext}(\mathcal{A},\mathcal{B})$ and hence $i_{\mathcal{A},\mathcal{B}}([\tau_1]_u) = [\tau]$, $[\tau_1]_u \in \operatorname{Ker}\bar{\Gamma}_{\mathcal{A},\mathcal{B}}$, i.e., $i_{\mathcal{A},\mathcal{B}}\colon \operatorname{Ker}\bar{\Gamma}_{\mathcal{A},\mathcal{B}} \to \operatorname{Ker}\Gamma_{\mathcal{A},\mathcal{B}}$ is surjective. Clearly, $\operatorname{Ran}(\Phi'_{\mathcal{A},\mathcal{B}} \subset \operatorname{Ker}\bar{\Gamma}_{\mathcal{A},\mathcal{B}}$. Thus, we get the exact sequence

$$0 \longrightarrow K_0(\mathcal{B})/H_1(K_0(\mathcal{A}),K_0(\mathcal{B})) \xrightarrow{\Phi'_{\mathcal{A},\mathcal{B}}} \operatorname{Ker}\bar{\Gamma}_{\mathcal{A},\mathcal{B}} \xrightarrow{i_{\mathcal{A},\mathcal{B}}} \operatorname{Ker}\Gamma_{\mathcal{A},\mathcal{B}} \longrightarrow 0$$

by Corollary 2.5. Therefore, we can deduce from following commutative diagram that $\bar{\kappa}$ is bijective.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{A}),K_0(\mathcal{B}))} \arrow[r, equal] \arrow[d] & \dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{A}),K_0(\mathcal{B}))} \arrow[d] & \\
\operatorname{Ext}_{\mathbb{Z}}(K_*(\mathcal{A}),[1_{\mathcal{A}}],K_*(\mathcal{B})) \arrow[d] & \operatorname{Ker}\bar{\Gamma}_{\mathcal{A},\mathcal{B}} \arrow[l, "\bar{\kappa}"'] \arrow[r] \arrow[d, "i_{\mathcal{A},\mathcal{B}}"'] & \operatorname{Ext}_u(\mathcal{A},\mathcal{B}) \arrow[d, "i_{\mathcal{A},\mathcal{B}}"'] \\
\operatorname{Ext}_{\mathbb{Z}}(K_*(\mathcal{A}),K_*(\mathcal{B})) \arrow[d] & \operatorname{Ker}\Gamma_{\mathcal{A},\mathcal{B}} \arrow[l, "\kappa"'] \arrow[r] \arrow[d] & \operatorname{Ext}(\mathcal{A},\mathcal{B}) \\
0 & 0 &
\end{tikzcd}$$

□

4 The half–exactness of $\operatorname{Ext}_u(\mathcal{A},\cdot)$

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be separable stable C^*–algebras and ϕ is a $*$–homomorphism of $\mathcal{B}_1$ to $\mathcal{B}_2$. Let $\phi_*^i\colon K_i(\mathcal{B}_1) \to K_i(\mathcal{B}_2)$ be the induced homomorphism of ϕ on $K_i(\mathcal{B}_1)$, $i = 0,1$. Then

$$\phi_*^i((H_1(K_0(\mathcal{A}),K_i(\mathcal{B}_1))) \subset H_1(K_0(\mathcal{A}),K_i(\mathcal{B}_2)),\ i = 0,1.$$

Define the homomorphism

$$\hat{\phi}_*\colon K_0(\mathcal{B}_1)/H_1(K_0(\mathcal{A}),K_0(\mathcal{B}_1)) \to K_0(\mathcal{B}_2)/H_1(K_0(\mathcal{A}),K_0(\mathcal{B}_2))$$

by $\hat{\phi}_*([x]_{\mathcal{A},\mathcal{B}_1}) = [\phi_*^0(x)]_{\mathcal{A},\mathcal{B}_2}$, $\forall\, x \in K_0(\mathcal{B}_1)$.

Let $\mathcal{B}_1, \mathcal{B}_2$ and ϕ be as above. If $\overline{\phi(\mathcal{B}_1)\mathcal{B}_2} = \mathcal{B}_2$ (especially, ϕ is surjective), then ϕ has a unique extension $\bar{\phi}$ from $M(\mathcal{B}_1)$ to $M(\mathcal{B}_2)$ such that $\bar{\phi}$ is strictly continuous and $\bar{\phi}(1_{M(\mathcal{B}_1)}) = 1_{M(\mathcal{B}_2)}$ by [11, Corollary 1.1.15]. Let $\pi_{\mathcal{B}_i}$ be the quotient map from $M(\mathcal{B}_i)$ onto $Q(\mathcal{B}_i)$, $i = 1,2$ and let $\hat{\phi}\colon Q(\mathcal{B}_1) \to Q(\mathcal{B}_2)$ be the unital $*$–homomorphism induced by $\bar{\phi}$ such that $\hat{\phi}\circ\pi_{\mathcal{B}_1} = \pi_{\mathcal{B}_2}\circ\bar{\phi}$. In this case, we can define homomorphisms $\phi_*\colon \operatorname{Ext}(\mathcal{A},\mathcal{B}_1) \to \operatorname{Ext}(\mathcal{A},\mathcal{B}_2)$ and $\phi_*^u\colon \operatorname{Ext}_u(\mathcal{A},\mathcal{B}_1) \to \operatorname{Ext}_u(\mathcal{A},\mathcal{B}_2)$ respectively by

$$\phi_*([\tau]) = [\hat{\phi}\circ\tau],\ \forall\,[\tau] \in \operatorname{Ext}(\mathcal{A},\mathcal{B}_1), \quad \phi_*^u([\tau]_u) = [\hat{\phi}\circ\tau]_u,\ \forall\,[\tau]_u \in \operatorname{Ext}_u(\mathcal{A},\mathcal{B}_1).$$

In general, according to [11, Lemma 1.3.19, Corollary 1.1.15], ϕ is homotopic to a quasi–unital $*$–homomorphism $\phi_0\colon \mathcal{B}_1 \to \mathcal{B}_2$ (i.e., $\overline{\phi_0(\mathcal{B}_1)\mathcal{B}_2} = p\mathcal{B}_2$ for some projection $p \in M(\mathcal{B}_2)$) and ϕ_0 has a unique extension $\bar{\phi}_0\colon M(\mathcal{B}_1) \to M(\mathcal{B}_2)$ with $p = \bar{\phi}_0(1_{M(\mathcal{B}_1)})$ such that $\bar{\phi}_0$ is strictly continuous. In this case, we define $\phi_*([\tau]) = [\hat{\phi}_0 \circ \tau]$, $\forall\,[\tau] \in \mathrm{Ext}(\mathcal{A}, \mathcal{B}_1)$ and $\phi^u_*([\tau]_u) = [\hat{\phi}_0 \circ \tau']_u$, $\forall\,\tau \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}_1))$, where

$$\tau'(a) = (\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))\big(\operatorname{diag}(\hat{\phi}_0 \circ \tau(a), 0) + \pi_{\mathcal{B}_2}(W)\operatorname{diag}(\tau_{\mathcal{A},\mathcal{B}_2}(a), 0)\pi_{\mathcal{B}_2}(W^*)\big)\begin{pmatrix}\pi_{\mathcal{B}_2}(u_1^*)\\ \pi_{\mathcal{B}_2}(u_2^*)\end{pmatrix},$$

$\forall\,a \in \mathcal{A}$ and $W \in \mathrm{M}_2(M(\mathcal{B}_2))$ such that $W^*W = \operatorname{diag}(1_{M(\mathcal{B}_2)}, 0)$, $WW^* = \operatorname{diag}(1_{M(\mathcal{B}_2)} - p, 1_{M(\mathcal{B}_2)})$, and u_1, u_2 are isometries in $M(\mathcal{B}_2)$ with $u_1u_1^* + u_2u_2^* = 1_{M(\mathcal{B}_2)}$. The ϕ_* above is well–defined (cf. [12, Remark 2.9]). In order to show ϕ^u_* is well–defined, we need following lemma.

Lemma 4.1. *Let $\sigma_t\colon \mathcal{A} \to M(\mathcal{B})$ be unital completely positive maps for all t in $[0,1]$ such that $t \mapsto \sigma_t(a)$ is strictly continuous for every a in $\mathcal{A}$ and $t \mapsto \sigma_t(ab) - \sigma_t(a)\sigma_t(b)$ is norm–continuous from $[0,1]$ to $\mathcal{B}$ for all $a, b \in \mathcal{A}$. Put $\tau_t = \pi \circ \sigma_t \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}))$, $\forall\,t \in [0,1]$. Suppose that $\mathcal{A} \in \mathcal{N}$. Then $[\tau_0]_u = [\tau_1]_u$ in $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$.*

Proof. Set $I = [0,1]$ and $I\mathcal{B} = \{f\colon I \to \mathcal{B}$ continuous$\}$. Let $\Lambda_i\colon I\mathcal{B} \to \mathcal{B}$ be $\Lambda_i(f) = f(i)$, $i = 0, 1$, $\forall\,f \in I\mathcal{B}$. Since $M(I\mathcal{B}) = \{f\colon I \to M(\mathcal{B})$ strictly continuous$\}$ and Λ_i is surjective, we have $\bar{\Lambda}_i(f) = f(i)$, $i = 0, 1$, $\forall\,f \in M(I\mathcal{B})$. Note that $\Lambda^i_{j*}\colon K_i(\mathcal{IB}) \to K_i(\mathcal{B})$ is isomorphic, $i, j = 0, 1$ and $\Lambda_{j*}\colon \mathrm{Ext}(\mathcal{A}, I\mathcal{B}) \to \mathrm{Ext}(\mathcal{A}, \mathcal{B})$ is also isomorphic by [1, Theorem 19.5.7], $j = 0, 1$ as $C_0((0,1], \mathcal{B})$, $C_0([0,1), \mathcal{B})$ are contractible and

$$0 \longrightarrow C_0((0,1], \mathcal{B}) \longrightarrow I\mathcal{B} \longrightarrow \mathcal{B} \longrightarrow 0, \quad 0 \longrightarrow C_0([0,1), \mathcal{B}) \longrightarrow I\mathcal{B} \longrightarrow \mathcal{B} \longrightarrow 0$$

are split exact.

Consider following diagram of exact sequences obtained by Corollary 2.5.

$$\begin{array}{ccccccccc}
0\longrightarrow & \dfrac{K_0(I\mathcal{B})}{H_1(K_0(\mathcal{A}), K_0(I\mathcal{B}))} & \xrightarrow{\Phi'_{\mathcal{A},I\mathcal{B}}} & \mathrm{Ext}_u(\mathcal{A}, I\mathcal{B}) & \xrightarrow{i_{\mathcal{A},I\mathcal{B}}} & \mathrm{Ext}(\mathcal{A}, I\mathcal{B}) & \xrightarrow{\rho_{\mathcal{A},I\mathcal{B}}} & H_1(K_0(\mathcal{A}), K_1(I\mathcal{B})) & \longrightarrow 0\\
& \Big\downarrow{\scriptstyle \hat{\Lambda}_{j*}} & & \Big\downarrow{\scriptstyle \Lambda^u_{j*}} & & \Big\downarrow{\scriptstyle \Lambda_{j*}} & & \Big\downarrow{\scriptstyle \Lambda^1_{j*}} & .\\
0\longrightarrow & \dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))} & \xrightarrow{\Phi'_{\mathcal{A},\mathcal{B}}} & \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}) & \xrightarrow{i_{\mathcal{A},\mathcal{B}}} & \mathrm{Ext}(\mathcal{A}, \mathcal{B}) & \xrightarrow{\rho_{\mathcal{A},\mathcal{B}}} & H_1(K_0(\mathcal{A}), K_1(\mathcal{B})) & \longrightarrow 0
\end{array}$$

According to the definitions of homomorphisms in the diagram, it is easy to verify that this diagram is commutative. Since $\hat{\Lambda}_{j*}$, Λ_{j*} and Λ^1_{j*} are all isomorphic, we have Λ^u_{j*} is isomorphic by 5–Lemma, $j = 0, 1$.

Let $R\colon \mathcal{B} \to I\mathcal{B}$ be given by $R(x)(t) = x$, $\forall\,x \in \mathcal{B}$ and $t \in I$. Then $\Lambda_j \circ R = \mathrm{id}_{\mathcal{B}}$ and $\overline{R(\mathcal{B})(I\mathcal{B})} = I\mathcal{B}$, $j = 0, 1$. So $R_*\colon \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}) \to \mathrm{Ext}_u(\mathcal{A}, I\mathcal{B})$ is well–defined and $\Lambda^u_{j*} \circ R^u_* = \mathrm{id}$ on $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B})$, $j = 0, 1$. Therefore, $\Lambda^u_{0*} = \Lambda^u_{1*}$ by above argument.

Now the map $\tilde{\sigma}(a)(t) = \sigma_t(a)$, $\forall\,a \in \mathcal{A}$ and $t \in I$ defines a unital completely map from $\mathcal{A}$ to $M(I\mathcal{B})$ with $\tilde{\sigma}(ab) - \tilde{\sigma}(a)\tilde{\sigma}(b) \in I\mathcal{B}$, $\forall\,a, b \in \mathcal{A}$. Put $\tilde{\tau} = \pi_{I\mathcal{B}} \circ \tilde{\sigma}$. Then $\tilde{\tau} \in \mathrm{Hom}_1(\mathcal{A}, Q(I\mathcal{B}))$ and $\Lambda^u_{j*}([\tilde{\tau}]_u) = [\tau_j]_u$, $j = 0, 1$, so that $[\tau_0]_u = [\tau_1]_u$. □

Proposition 4.2. *Let $\mathcal{A} \in \mathcal{N}$ be a unital separable nuclear C^*–algebra and $\mathcal{B}_1, \mathcal{B}_2$ be separable stable C^*–algebras. Let $\phi\colon \mathcal{B}_1 \to \mathcal{B}_2$ be a $*$–homomorphism. We have*

(1) $\mathrm{Ext}_u(\cdot, \mathcal{B})$ *is homotopic invariant for first variable in the class of separable nuclear algebras belonging to $\mathcal{N}$;*

(2) $\phi^u_*\colon \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}_1) \to \mathrm{Ext}_u(\mathcal{A}, \mathcal{B}_2)$ *given above is well–defined;*

(3) $\mathrm{Ext}_u(\mathcal{A}, \cdot)$ *is homotopic invariant for second variable in the class of separable, stable C^*–algebras;*

(4) $\phi_* \circ i_{\mathcal{A},\mathcal{B}_1} = i_{\mathcal{A},\mathcal{B}_2} \circ \phi^u_*$, $\phi^u_* \circ \Phi'_{\mathcal{A},\mathcal{B}_1} = \Phi'_{\mathcal{A},\mathcal{B}_2} \circ \hat{\phi}_*$.

Proof. (1) Let $\mathcal{A}_1, \mathcal{A}_2$ be unital separable nuclear C^*–algebras which is in $\mathcal{N}$. Let α_1, α_2 be unital $*$–homomorphisms from $\mathcal{A}_1$ to $\mathcal{A}_2$. Suppose that there is a path of unital $*$–homomorphism ρ_t from $\mathcal{A}_1$ to $\mathcal{A}_2$ for all $t \in [0,1]$ such that $t \mapsto \rho_t(a)$ is continuous from $[0,1]$ to $\mathcal{A}_2$ for ever $a \in \mathcal{A}_1$ and $\rho_0 = \alpha_1$, $\rho_1 = \alpha_2$.

Let $\tau \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}))$. Then there is a unital completely positive map $\sigma\colon \mathcal{A}_2 \to M(\mathcal{B})$ such that $\tau = \pi \circ \tau$. Set $\sigma_t = \sigma \circ \rho_t$ and $\tau_t = \pi \circ \sigma_t$, $\forall\, t \in [0,1]$. Obviously, $\{\sigma_t | \, t \in [0,1]\}$ satisfies the conditions given in Lemma 4.1. Thus, $\alpha_1^*([\tau]_u) = [\tau \circ \rho_0]_u = [\tau \circ \rho_1]_u = \alpha_2^*([\tau]_u)$.

(2) If ϕ is homotopic to another quasi–unital $*$–homomorphism ϕ', then by [11, Lemma 3.1.15], there is a path λ_t of $*$–homomorphism from $M(\mathcal{B}_1)$ to $M(\mathcal{B}_2)$, $\forall\, t \in [0,1]$ such that

(a) $t \mapsto \lambda_t(x)$ is strictly continuous for every $x \in M(\mathcal{B}_1)$;

(b) $t \mapsto \lambda_t(b)$ is norm–continuous from $[0,1]$ to $\mathcal{B}_2$ for any $b \in \mathcal{B}_1$;

(c) $\lambda_0 = \bar{\phi}_0$, $\lambda_1 = \bar{\phi}'$.

Put $\hat{p}(t) = \lambda_t(1_{M(\mathcal{B}_1)})$. Then $\hat{p}$ is a projection in $M(I\mathcal{B}_2)$ and hence there ia partial isometry $\hat{w}$ in $\mathrm{M}_2(M(I\mathcal{B}_2))$ such that

$$\hat{w}^*\hat{w} = \mathrm{diag}\,(1_{M(I\mathcal{B}_2)}, 0) \text{ and } \hat{w}\hat{w}^* = \mathrm{diag}\,(1_{M(I\mathcal{B}_2)} - \hat{p}, 1_{M(I\mathcal{B}_2)}).$$

Simple computation shows that W (resp. $\hat{w}$) has the form $W = \begin{pmatrix} w_1 & 0 \\ w_2 & 0 \end{pmatrix}$ (resp. $\hat{w} = \begin{pmatrix} \hat{w}_1 & 0 \\ \hat{w}_2 & 0 \end{pmatrix}$) with

$$w_1 w_1^* = 1_{M(\mathcal{B}_2)} - p,\ w_1 w_2^* = 0,\ w_2 w_2^* = 1_{M(\mathcal{B}_2)},\ w_1^* w_1 + w_2^* w_2 = 1_{M(\mathcal{B}_2)};$$
$$\hat{w}_1 \hat{w}_1^* = 1_{M(\mathcal{B}_2)} - \hat{p},\ \hat{w}_1 \hat{w}_2^* = 0,\ \hat{w}_2 \hat{w}_2^* = 1_{M(\mathcal{B}_2)},\ \hat{w}_1^* \hat{w}_1 + \hat{w}_2^* \hat{w}_2 = 1_{M(\mathcal{B}_2)}.$$

Set $v = w_1^* \hat{w}_1(0) + w_2^* \hat{w}_2(0)$. Then v is unitary in $M(\mathcal{B}_2)$ and $w_1 v = \hat{w}_1(0)$, $w_2 v = \hat{w}_2(0)$. Let $v(t)$ be a strictly continuous path in $U(M(\mathcal{B}_2))$ ($t \in [0,1]$) such that $v(0) = v$, $v(1) = 1_{M(\mathcal{B}_2)}$. Set

$$\hat{W}_i(t) = \begin{cases} \hat{w}_i(0) v^*(2t) & 0 \le t \le 1/2 \\ \hat{w}_i(2t-1) & 1/2 \le t \le 1 \end{cases}, \quad i = 1, 2, \quad \hat{W} = \begin{pmatrix} \hat{W}_1 & 0 \\ \hat{W}_2 & 0 \end{pmatrix}.$$

Then $\hat{W} \in M(I\mathcal{B}_2)$ with $\hat{W}(0) = W$. Set

$$\hat{q}(t) = \begin{cases} p & 0 \le t \le 1/2 \\ \hat{p}(2t-1) & 1/2 \le t \le 1 \end{cases}, \quad \hat{\lambda}_t = \begin{cases} \bar{\phi}_0 & 0 \le t \le 1/2 \\ \lambda_{2t-1} & 1/2 \le t \le 1 \end{cases}.$$

Clearly, $\hat{q}(t) = \hat{\lambda}_t(1_{M(\mathcal{B}_2)})$ and $\hat{\lambda}_t$ satisfies Condition (a), (b) and (c), and

$$\hat{W}^*\hat{W} = \operatorname{diag}(1_{M(\mathcal{B}_2)}, 0), \quad \hat{W}\hat{W}^* = \operatorname{diag}(1_{M(\mathcal{B}_2)} - \hat{q}, 1_{M(\mathcal{B}_2)}).$$

Let $q = \bar{\phi}'(1_{M(\mathcal{B}_1)})$. Pick $W' \in \mathrm{M}_2(M(\mathcal{B}_2))$ such that $W'W'^* = \operatorname{diag}(1_{M(\mathcal{B}_2)}, 0)$, $W'^*W' = \operatorname{diag}(1_{M(\mathcal{B}_2)} - q, 1_{M(\mathcal{B}_2)})$. If $W' \neq \hat{W}(1)$, using above method, we can choose $\hat{\lambda}_t$ and $\hat{W}$ such that $\hat{W}(1) = W'$.

Let $\tau \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}_1))$ with $\tau = \pi_{\mathcal{B}_1} \circ \sigma$ for some unital completely positive map $\sigma\colon \mathcal{A} \to M(\mathcal{B}_2)$ and let $\tau_{\mathcal{A},\mathcal{B}_2} = \pi_{\mathcal{B}_2} \circ \psi$ for some unital $*$–homomorphism $\psi\colon \mathcal{A} \to M(\mathcal{B}_2)$. Set

$$\sigma_t(a) = (u_1, u_2)(\operatorname{diag}(\hat{\lambda}_t \circ \sigma(a), 0) + \hat{W}(t)\operatorname{diag}(\psi(a), 0)\hat{W}^*(t))(u_1, u_2)^T, \ \forall\, a \in \mathcal{A}.$$

Note that $\sigma(ab) - \sigma(a)\sigma(b) \in \mathcal{B}_1$, $\forall\, a, b \in \mathcal{A}$ and $\hat{\lambda}_t$ satisfies Condition (a), (b) and (c). So we have $[\pi_{\mathcal{B}_2} \circ \sigma_0]_u = [\pi_{\mathcal{B}_2} \circ \sigma_1]_u$ by Lemma 4.1, that is, ϕ^u_* is well–defined.

(3) By (2), we can assume that ϕ_1, $\phi_2\colon \mathcal{B}_1 \to \mathcal{B}_2$ are quasi–unital $*$–homomorphisms which are homotopic. Using the same methods as in the proof of (2), we have $\phi^u_{1*} = \phi^u_{2*}$.

(4) Put $r = \begin{pmatrix} \pi_{\mathcal{B}_2}(p) & 0 \\ 0 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} r & 1_2 - r \\ 1_2 - r & r \end{pmatrix} \in U_0(\mathrm{M}_4(Q(\mathcal{B}_2)))$, where 1_2 is the unit of $\mathrm{M}_2(Q(\mathcal{B}_2))$. Then for any $\tau \in \mathrm{Hom}_1(\mathcal{A}, Q(\mathcal{B}_1))$ and any $a \in \mathcal{A}$,

$$\begin{aligned} \operatorname{diag}\Big(\begin{pmatrix} \hat{\phi}_0 \circ \tau(a) & \\ & 0 \end{pmatrix} + \pi^{(2)}_{\mathcal{B}_2}(W)\begin{pmatrix} \tau_{\mathcal{A},\mathcal{B}_2}(a) & \\ & 0 \end{pmatrix}\pi^{(2)}_{\mathcal{B}_2}(W^*), 0\Big) \\ = U\begin{pmatrix} \hat{\phi}_0 \circ \tau(a) & \\ & \pi^{(2)}_{\mathcal{B}_2}(W)\begin{pmatrix} \tau\mathcal{A}, \mathcal{B}_2(a) & \\ & 0 \end{pmatrix}\pi^{(2)}_{\mathcal{B}_2}(W^*) \end{pmatrix} U^*, \end{aligned}$$

where $\pi^{(n)}_{\mathcal{B}_2}$ represents the induced homomorphism of $\pi_{\mathcal{B}_2}$ on $\mathrm{M}_n(M(\mathcal{B}_2))$. Put

$$\bar{W} = \begin{pmatrix} W & 1_2 - WW^* \\ 1_2 - W^*W & W^* \end{pmatrix} \in U(\mathrm{M}_4(M(\mathcal{B}_2))).$$

Since $1_2 - W^*W = \operatorname{diag}(0, 1)$, it follows that for every $a \in \mathcal{A}$,

$$\operatorname{diag}\Big(\pi^{(2)}_{\mathcal{B}_2}(W)\begin{pmatrix} \tau_{\mathcal{A},\mathcal{B}_2}(a) & \\ & 0 \end{pmatrix}\pi^{(2)}_{\mathcal{B}_2}(W^*), 0\Big) = \pi^{(4)}_{\mathcal{B}_2}(\bar{W})\operatorname{diag}\Big(\begin{pmatrix} \tau_{\mathcal{A},\mathcal{B}_2}(a) & \\ & 0 \end{pmatrix}, 0\Big)\pi^{(4)}_{\mathcal{B}_2}(\bar{W}^*).$$

Therefore, $i_{\mathcal{A},\mathcal{B}_2} \circ \phi^u_*([\tau]_u) = [\hat{\phi}_0 \circ \tau] = \phi_* \circ i_{\mathcal{A},\mathcal{B}_1}([\tau]_u)$ in $\mathrm{Ext}_u(\mathcal{A}, \mathcal{B}_2)$.

Let $x \in K_0(\mathcal{B}_1)$. Then there is $v \in U(Q(\mathcal{B}_1))$ such that $\partial_0([v]) = x$ and $\Phi'_{\mathcal{A},\mathcal{B}_1}([x]_{\mathcal{A},\mathcal{B}_1}) = [Ad_v \circ \tau_{\mathcal{A},\mathcal{B}_1}]_u$. Let $\tau_{\mathcal{A},\mathcal{B}_1} = \pi_{\mathcal{B}_1} \circ \psi_0$ for some unital $*$–homomorphism $\psi_0\colon \mathcal{A} \to M(\mathcal{B}_1)$ and set

$$\tilde{\psi}(a) = (u_1, u_2)\Big(\begin{pmatrix} \bar{\phi}_0 \circ \psi_0(a) & \\ & 0 \end{pmatrix} + W\begin{pmatrix} \psi_0(a) & \\ & 0 \end{pmatrix} W^*\Big)(u_1, u_2)^T, \ \forall\, a \in \mathcal{A},$$

$$\tilde{v} = (\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))\operatorname{diag}(\hat{\phi}_0(v) + 1_{Q(\mathcal{B}_2)} - \pi_{\mathcal{B}_2}(p), 1_{Q(\mathcal{B}_2)})(\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))^T \in U(Q(\mathcal{B}_2)).$$

Then $\tilde{\psi}\colon \mathcal{A} \to M(\mathcal{B}_2)$ is a unital $*$–homomorphism and

$$\phi_*^u \circ \Phi'_{\mathcal{A},\mathcal{B}_1}([x]_{\mathcal{A},\mathcal{B}_1}) = [Ad_{\tilde{v}} \circ (\pi_{B_2} \circ \tilde{\psi})]_u = [Ad_{\tilde{v}} \circ \tau_{\mathcal{A},\mathcal{B}_2}]_u,$$

by Lemma 2.3. Noting that $[\tilde{v}] = [\hat{\phi}_0(v) + 1_{Q(\mathcal{B}_2)} - \pi_{\mathcal{B}_2}(p)]$ in $K_1(Q(\mathcal{B}_2))$ by Lemma 2.1 and

$$\partial_0([\hat{\phi}_0(v) + 1_{Q(\mathcal{B}_2)} - \pi_{\mathcal{B}_2}(p)]) = [\phi_{0*}^0(x)] = [\phi_*^0(x)],$$

We have

$$\Phi'_{\mathcal{A},\mathcal{B}_2} \circ \hat{\phi}_*([x]_{\mathcal{A},\mathcal{B}_1}) = \Phi'_{\mathcal{A},\mathcal{B}_2}([\phi_*^0(x)]_{\mathcal{A},\mathcal{B}_2}) = [Ad_{\tilde{v}} \circ \tau_{\mathcal{A},\mathcal{B}_2}]_u = \phi_*^u \circ \Phi'_{\mathcal{A},\mathcal{B}_1}([x]_{\mathcal{A},\mathcal{B}_1}).$$

$\square$

Remark 4.3. *In the definition of ϕ_*^u, we can replace $\tau_{\mathcal{A},\mathcal{B}_2}$ by any unital extension $\mu\colon \mathcal{A} \to Q(\mathcal{B}_2)$. Let ϕ_0, W and u_1, u_2 be as above and let $\mu_0\colon \mathcal{A} \to Q(\mathcal{B}_2)$ be a unital trivial extension. Put*

$$\begin{aligned}
\tau_0'(a) &= \operatorname{diag}(\hat{\phi}_0 \circ \tau(a), 0) + \pi_{\mathcal{B}_2}(W) \operatorname{diag}(\tau_{\mathcal{A},\mathcal{B}_2}(a), 0)\pi_{\mathcal{B}_2}(W^*),\\
\tau_0''(a) &= \operatorname{diag}(\hat{\phi}_0 \circ \mu_0(a), 0) + \pi_{\mathcal{B}_2}(W) \operatorname{diag}(\mu(a), 0)\pi_{\mathcal{B}_2}(W^*),\\
\tau_1'(a) &= \operatorname{diag}(\hat{\phi}_0 \circ \tau(a), 0) + \pi_{\mathcal{B}_2}(W) \operatorname{diag}(\mu(a), 0)\pi_{\mathcal{B}_2}(W^*),\\
\tau_1''(a) &= \operatorname{diag}(\hat{\phi}_0 \circ \mu_0(a), 0) + \pi_{\mathcal{B}_2}(W) \operatorname{diag}(\tau_{\mathcal{A},\mathcal{B}_2}(a), 0)\pi_{\mathcal{B}_2}(W^*),\\
\tau'(a) &= (\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))\tau_0'(a)(\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))^T,\\
\tau''(a) &= (\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))\tau_1'(a)(\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))^T,
\end{aligned}$$

$\forall\, a \in \mathcal{A}$. Since $U \operatorname{diag}(\tau_0'(a), \tau_0''(a))U^* = \operatorname{diag}(\tau_1'(a), \tau_1''(a))$, $\forall\, a \in \mathcal{A}$, where U is given in the proof of Proposition 4.2 (4) and

$$(\pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))\tau_0''(\cdot)\begin{pmatrix}\pi_{\mathcal{B}_2}(u_1^*)\\ \pi_{\mathcal{B}_2}(u_2^*)\end{pmatrix},\ \pi_{\mathcal{B}_2}(u_1), \pi_{\mathcal{B}_2}(u_2))\tau_1''(\cdot)\begin{pmatrix}\pi_{\mathcal{B}_2}(u_1^*)\\ \pi_{\mathcal{B}_2}(u_2^*)\end{pmatrix}$$

are all unital trivial extensions, we obtain that $[\tau']_u = [\tau'']_u$ in $\operatorname{Ext}_u(\mathcal{A}, \mathcal{B}_2)$.

Let $\mathcal{I}$ be a closed ideal of $\mathcal{B}$ and let $\Lambda\colon \mathcal{B} \to \mathcal{B}/\mathcal{I} = \mathcal{D}$ be the canonical homomorphism. Then $\mathcal{I}$ and $\mathcal{D}$ are all stable by [13, Corollary 2.3 (ii)].

Theorem 4.4. *Let the short exact sequence $0 \longrightarrow \mathcal{I} \xrightarrow{j} \mathcal{B} \xrightarrow{\Lambda} \mathcal{D} \longrightarrow 0$ be given as above. Suppose that $\mathcal{A} \in \mathcal{N}$ is a unital separable unclear C^*–algebra and*

(i) *there is a completely positive map $\Psi\colon \mathcal{D} \to \mathcal{B}$ such that $\Lambda \circ \Psi = \mathrm{id}_{\mathcal{D}}$;*

(ii) $\operatorname{Ran}(\hat{j}_*) = \operatorname{Ker}\hat{\Lambda}_*$;

(iii) *$j_*^1\colon H_1(K_0(\mathcal{A}), K_1(\mathcal{I})) \to H_1(K_0(\mathcal{A}), K_1(\mathcal{B}))$ is injective.*

Then

$$\mathrm{Ext}_u(\mathcal{A},\mathcal{I}) \xrightarrow{j_*^u} \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \xrightarrow{\Lambda_*^u} Ext_u(\mathcal{A},\mathcal{D}) \tag{4.1}$$

is exact in the middle. Especially, if the ψ *in Condition* (i) is a $*$–homomorphism, then j_*^u is injective, Λ_*^u is surjective in (4.1) and (4.1) is split exact.

Proof. Let $j_0\colon \mathcal{I}\to\mathcal{B}$ be a quasi–unital $*$–homomorphism with $\bar{j}_0(1_{M(\mathcal{I})}) = p \in M(\mathcal{B})$ such that j_0 is homotopic to j. Then $\Lambda\circ j_0\colon \mathcal{I}\to\mathcal{D}$ is homotopic to $\Lambda\circ j=0$. So $(\Lambda\circ j_0)_*^u=0$ by Proposition 4.2. Let $\tau\in \mathrm{Hom}_1(\mathcal{A},Q(\mathcal{I}))$. Then $\Lambda_*^u\circ j_*^u([\tau]_u)=[\hat{\Lambda}\circ\tau']_u$, where

$$\tau'(a)=(\pi(u_1),\pi(u_2))\Big(\begin{pmatrix}\hat{j}_0\circ\tau(a) & \\ & 0\end{pmatrix}+\pi^{(2)}(W)\begin{pmatrix}\tau_{\mathcal{A},\mathcal{B}}(a) & \\ & 0\end{pmatrix}\pi^{(2)}(W^*)\Big)(\pi(u_1),\pi(u_2))^T,$$

$\forall\, a\in\mathcal{A}$and $W^*W=\mathrm{diag}\,(1,0)$, $WW^*=\mathrm{diag}\,(1-p,1)$. Set $v_i=\pi(u_i)$, $i=1,2$, $p_0=\bar{\Lambda}(p)$ and $W_0=\bar{\Lambda}(W)$. Since v_1,v_2 are isometries with $v_1v_1^*+v_2v_2^*=1$ and $\hat{\Lambda}\circ\tau_{\mathcal{A},\mathcal{B}}\colon\mathcal{A}\to Q(\mathcal{D})$ is a trivial unital extension and $W_0^*W_0=\mathrm{diag}\,(1,0)$, $W_0W_0^*=\mathrm{diag}\,(1-p_0,1)$ in $\mathrm{M}_2(M(\mathcal{D}))$. Noting that $\overline{\Lambda\circ j_0}=\bar{\Lambda}\circ\bar{j}_0$, we have $\widehat{\Lambda\circ j_0}=\hat{\Lambda}\circ\hat{j}_0$. Therefore, $\Lambda_*^u\circ j_*^u([\tau]_u)=(\Lambda\circ j_0)_*^u([\tau]_u)=0$, i.e., $\mathrm{Ran}\,(j_*^u)\subset\mathrm{Ker}\,\Lambda_*^u$.

In order to prove $\mathrm{Ker}\,\Lambda_*^u\subset\mathrm{Ran}\,(j_*^u)$, we consider following diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow\\
\dfrac{K_0(\mathcal{I})}{H_1(K_0(\mathcal{A}),K_0(\mathcal{I})} & \xrightarrow{\hat{j}_*} & \dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{A}),K_0(\mathcal{B})} & \xrightarrow{\hat{\Lambda}_*} & \dfrac{K_0(\mathcal{D})}{H_1(K_0(\mathcal{A}),K_0(\mathcal{D})}\\
{\scriptstyle\Phi'_{\mathcal{A},\mathcal{I}}}\downarrow & & {\scriptstyle\Phi'_{\mathcal{A},\mathcal{B}}}\downarrow & & {\scriptstyle\Phi'_{\mathcal{A},\mathcal{D}}}\downarrow\\
\mathrm{Ext}_u(\mathcal{A},\mathcal{I}) & \xrightarrow{j_*^u} & \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) & \xrightarrow{\Lambda_*^u} & \mathrm{Ext}_u(\mathcal{A},\mathcal{D})\\
{\scriptstyle i_{\mathcal{A},\mathcal{I}}}\downarrow & & {\scriptstyle i_{\mathcal{A},\mathcal{B}}}\downarrow & & {\scriptstyle i_{\mathcal{A},\mathcal{D}}}\downarrow\\
\mathrm{Ext}(\mathcal{A},\mathcal{I}) & \xrightarrow{j_*} & \mathrm{Ext}(\mathcal{A},\mathcal{B}) & \xrightarrow{\Lambda_*} & \mathrm{Ext}(\mathcal{A},\mathcal{D})\\
{\scriptstyle\rho_{\mathcal{A},\mathcal{I}}}\downarrow & & {\scriptstyle\rho_{\mathcal{A},\mathcal{B}}}\downarrow & & {\scriptstyle\rho_{\mathcal{A},\mathcal{D}}}\downarrow\\
H_1(K_0(\mathcal{A}),K_1(\mathcal{I})) & \xrightarrow{j_*^1} & H_1(K_0(\mathcal{A}),K_0(\mathcal{B})) & \xrightarrow{\Lambda_*^1} & H_1(K_0(\mathcal{A}),K_0(\mathcal{D}))\\
\downarrow & & \downarrow & & \downarrow\\
0 & & 0 & & 0
\end{array}\quad . \tag{4.2}$$

In (4.2), three columns are exact by Corollary 2.5, the first row is exact by Condition (ii) and the third row is exact by [1, Theorem 19.5.7]. It is easy to check that following two diagrams

$$\begin{array}{ccc}
K_0(Q(\mathcal{I})) & \xrightarrow{\hat{j}_{0*}^0} & K_0(Q(\mathcal{B}))\\
{\scriptstyle\partial_1}\downarrow & & {\scriptstyle\partial_1}\downarrow\\
K_1(\mathcal{I}) & \xrightarrow{j_{0*}^1} & K_1(\mathcal{B})
\end{array}\quad ,\quad
\begin{array}{ccc}
K_0(Q(\mathcal{B})) & \xrightarrow{\hat{\Lambda}_*^0} & K_0(Q(\mathcal{D}))\\
{\scriptstyle\partial_1}\downarrow & & {\scriptstyle\partial_1}\downarrow\\
K_1(\mathcal{B}) & \xrightarrow{\Lambda_*^1} & K_1(\mathcal{D})
\end{array}$$

are commutative. Thus, $\rho_{\mathcal{A},\mathcal{B}}\circ j_*=j_*^1\circ\rho_{\mathcal{A},\mathcal{I}}$, $\Lambda_*^1\circ\rho_{\mathcal{A},\mathcal{B}}=\rho_{\mathcal{A},\mathcal{D}}\circ\Lambda_*$. Finally, the diagram (4.2) is commutative by Proposition 4.2 (4). We can deduce the assertion from (4.2). $\square$

Corollary 4.5. *Let $\mathcal{I}$, $\mathcal{B}$, $\mathcal{D}$ and j, Λ be as in Theorem 4.4. Assume that $\mathcal{D}$ is contractible and there is a completely positive map $\Psi\colon \mathcal{D} \to \mathcal{B}$ such that $\Lambda \circ \Psi = \mathrm{id}_{\mathcal{D}}$. Then $j_*^u\colon \mathrm{Ext}_u(\mathcal{A},\mathcal{I}) \to \mathrm{Ext}_u(\mathcal{A},\mathcal{B})$ is isomorphic.*

Proof. The assumptions indicate that $\hat{j}_*$, j_* and j_*^1 are all isomorphic. By using Theorem 4.4, Proposition 4.2 (3) to the commutative diagram (4.2), we can obtain the assertion. $\square$

Corollary 4.6. *Let $\mathcal{A} \in \mathcal{N}$ be a unital separable nuclear C^*–algebra and $\mathcal{B}$ be a separable nuclear stable C^*–algebra. Then $\mathrm{Ext}_u(\mathcal{A}, S^2\mathcal{B}) \cong \mathrm{Ext}_u(\mathcal{A},\mathcal{B})$, where $S^2\mathcal{B} = C_0(\mathbb{R}^2) \otimes \mathcal{B}$.*

Proof. Let S be the unilateral shift on l^2 and let $T_0 = C^*(\mathrm{S} - I_{l^2})$ be the C^*–subalgebra in $B(l^2)$ generated by $\mathrm{S} - I_{l^2}$. Then we have a short exact sequence:

$$0 \longrightarrow \mathcal{B} \xrightarrow{j} T_0 \otimes \mathcal{B} \xrightarrow{q} S\mathcal{B} \longrightarrow 0 \tag{4.3}$$

and also have $K_i(T_0 \otimes \mathcal{B}) \cong 0$, $i = 0, 1$ (cf. [14]). Set

$$C_q = \{(x, f) \in (T_0 \otimes \mathcal{B}) \oplus C_0([0, 1), S\mathcal{B}) |\, q(x) = f(0)\}.$$

Then we have following exact sequences of C^*–algebras

$$0 \longrightarrow \mathcal{B} \xrightarrow{e} C_q \longrightarrow C_0([0, 1), SB) \longrightarrow 0 \tag{4.4}$$

$$0 \longrightarrow S^2\mathcal{B} \xrightarrow{i} C_q \longrightarrow T_0 \otimes \mathcal{B} \longrightarrow 0. \tag{4.5}$$

Since $C_0([0, 1), S\mathcal{B})$ is contractible, applying Corollary 4.5 to (4.4), we get that $e_*^u\colon \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) \to \mathrm{Ext}_u(\mathcal{A}, C_q \otimes \mathcal{K})$ is isomorphic. We also obtain that $e_*\colon \mathrm{Ext}(\mathcal{A},\mathcal{B}) \to \mathrm{Ext}(\mathcal{A}, C_q \otimes \mathcal{K})$ and $e_*^i\colon K_i(\mathcal{B}) \to K_i(C_q)$ are all isomorphic, $i = 0, 1$. Set

$$\partial_u = (e_*^u)^{-1} \circ i_*^u,\ \hat{\partial} = (\hat{e}_*)^{-1} \circ \hat{i}_*,\ \partial = (e_*)^{-1} \circ i_*,\ \partial^1 = (e_*^1)^{-1} \circ i_*^1.$$

Then we have following commutative diagram of exact sequences by Proposition 4.2:

$$\begin{array}{ccccccccc}
0 \to & \dfrac{K_0(\mathcal{B})}{H_1(K_0(\mathcal{A}), K_0(\mathcal{B}))} & \xrightarrow{\Phi'_{\mathcal{A},\mathcal{B}}} & \mathrm{Ext}_u(\mathcal{A},\mathcal{B}) & \xrightarrow{i_{\mathcal{A},\mathcal{B}}} & \mathrm{Ext}(\mathcal{A},\mathcal{B}) & \xrightarrow{\rho_{\mathcal{A},\mathcal{B}}} & H_1(K_0(\mathcal{A}), K_1(\mathcal{B})) & \to 0 \\
& \hat{\partial}\Big\uparrow & & \partial_u\Big\uparrow & & \partial\Big\uparrow & & \partial^1\Big\uparrow & \\
0 \to & \dfrac{K_0(S^2\mathcal{B})}{H_1(K_0(\mathcal{A}), K_0(S^2\mathcal{B}))} & \xrightarrow{\Phi'_{\mathcal{A},S^2\mathcal{B}}} & \mathrm{Ext}_u(\mathcal{A},S^2\mathcal{B}) & \xrightarrow{i_{\mathcal{A},S^2\mathcal{B}}} & \mathrm{Ext}(\mathcal{A},S^2\mathcal{B}) & \xrightarrow{\rho_{\mathcal{A},S^2\mathcal{B}}} & H_1(K_0(\mathcal{A}), K_1(S^2\mathcal{B})) & \to 0
\end{array}.$$

Since $K_i(T_0 \otimes \mathcal{B}) \cong 0$, $i = 0, 1$, it follows from (4.5) that $\hat{i}_*$, i_* and i_*^1 are all isomorphic. Thus, $\hat{\partial}$, ∂ and ∂^1 are isomorphic in above commutative diagram and consequently, ∂_u is isomorphic. $\square$

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